SINGULAR SCHAEFFER–SALEM MEASURES ON $[0,1]$ OF DYNAMICAL SYSTEM ORIGIN

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Abstract. We study a class of dynamical systems given by measure preserving actions of the group $\mathbb{Z}^d$ or $\mathbb{R}^d$ and generating a set of spectral measures σ with an extremal rate of the Fourier coefficient decay: $\widehat{\sigma}(n) = O(|n|^{-1/2+\varepsilon})$ for any $\varepsilon > 0$. Singular measures with this property are investigated in works due to Wiener and Wintner, Schaeffer, Salem, Ivashev-Musatov, Zygmund et al. Thus, the discovered effect provides a new construction of singular distributions of Schaeffer–Salem type on the torus $\mathbb{T}^d$ and in the space $\mathbb{R}^d$.

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1. Constructions and properties of Borel measures on $[0,1]$

In this work we investigate a class of dynamical systems generating spectral measures characterized by fast Fourier coefficient decay. We start with a well-known construction due to Riesz [17]. He proposed to consider a formal infinite product

$$\prod_{n=1}^{\infty} \big(1 + a_n \cos(\omega_n x + \phi_n)\big), \tag{1}$$

where $\omega_n \in 2\pi\mathbb{Z}$ is an increasing sequence, $0 < a_n \le 1$ and $\phi_n \in \mathbb{R}$. It is well known that for a certain choice of parameters a_n, ω_n and ϕ_n, for example, if $\omega_{n+1}/\omega_n \ge q > 3$ and $\sum_n a_n^2 = \infty$, this product represents a singular measure on $[0,1]$ (see [27], § 7). We understand this statement as follows. The finite products

$$\rho_N(x) = \prod_{n\le N} \big(1 + a_n \cos(\omega_n x + \phi_n)\big)$$

are interpreted as densities of probability measures on $[0,1]$, and we have convergence

$$\rho_N(x)\, ds \to d\sigma \qquad n \to \infty$$

in the weak topology, where σ is a measure on $[0,1]$. The infinite products (1) today refered to as *classical Riesz products* as well as generalized Riesz products

$$\prod_{n=1}^{\infty} P_n(z), \qquad P_n(z) = \sum_{k=0}^{q_n-1} c_{n,k}\, z^k, \quad z \in \mathbb{C}, \quad |z| = 1,$$

provide an important construction of singular measures broadly applied in analysis and dynamical systems (see, [5], [1], [2], [4], [22]).

Let us denote $\mathscr{M}([0,1])$ or simply $\mathscr{M}$ the class of all Borel probability measures on $[0,1]$. A measure ν is *absolutely continuous* with respect to another measure μ, notation: $\nu \ll \mu$, if $\nu = p(x)\mu$, where $p(x)$ is a certain density, $p(x) \in L^1(\mu)$. The relation $\nu \ll \mu$ is a partial order on $\mathscr{M}$. The measures μ and ν in the class $\mathscr{M}$ are called *mutual singular*, $\mu \perp \nu$, if there

exists a Borel set E such that $\mu(E) = \nu(E^c) = 1$, where E^c is the complement to the set E. Recall that any Borel measure σ on a segment in the real line is uniquely expanded in a sum

$$\sigma = \sigma_d + \sigma_s + \sigma_{ac}, \qquad \sigma_s \perp \lambda, \quad \sigma_{ac} = p(x)\lambda, \tag{2}$$

of discrete (a sum of atoms), purely singular and absolutely continuous components, where λ is the normalized Lebesgue measure on the segment. Since every measure $\sigma \in \mathscr{M}([0,1))$ can be considered as a measure on $\mathbb{R}$ one can define the *Fourier transform* of σ by the formula

$$\mathsf{F}[\sigma](t) = \int_0^{2\pi} e^{-2\pi i\, tx}\, d\sigma(x), \qquad t \in \mathbb{R}.$$

For probability measures on $\mathbb{R}$ the following general observation holds (see [23], ch. 2, § 12). If the Fourier transform of σ belongs to $L^1(\mathbb{R})$, for example, if $\mathsf{F}[\sigma](t) = O(t^{-1-\alpha})$, $\alpha > 0$, then σ in absolutely continuous. At the same time, any measure $\sigma \in \mathscr{M}([0,1))$ as a measure on the compact group $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq [0,1)$ generates the sequence of *Fourier coefficients* $\widehat{\sigma}(n)$ supported on the dual group $\widehat{\mathbb{T}} = \mathbb{Z}$. Note that

$$\widehat{\sigma}(n) = \mathsf{F}[\sigma](n) = \int_0^{2\pi} e^{-2\pi i\, nx}\, d\sigma(x), \qquad n \in \mathbb{Z}.$$

Suppose that $\widehat{\sigma} \in l^2(\mathbb{Z})$. Then the Fourier series $\sum_n \widehat{\sigma}(n)\, e^{2\pi i\, nt}$ converges in the space $L^2(\mathbb{T})$ to some function $p(x)$. Using Cauchy–Schwarz inequality $\|p\|_1 = \langle |p|, 1\rangle \le \|p\|_2$ we see that $p(x)$ is a density of some measure $p(x)\, dx \in \mathscr{M}(\mathbb{R})$, and $\widehat{p}(n) = \widehat{\sigma}(n)$ (eg. see [26], § 1). Furhter, notice that any sequence $c_n = O(n^{-1/2-\alpha})$, $\alpha > 0$, is square summable.

In the case of singular measure σ, as a rule, we deal with a divergent series $\sum_n \widehat{\sigma}(n)\, z^n$. Understanding analytic properties of a singular measure σ, when we know certain combinatorial properties of the sequence $\widehat{\sigma}(n)$, becomes a very complicated problem. Louzin [9] constructed the first example of power series $\sum_n c_n\, z^n$ with $c_n \to 0$ divergent everywhere on the unit circle $|z| = 1$. Further, Neder [13] proved that any series $\sum_n c_n\, z^n$ with the property $\sum_n |c_n|^2 = \infty$ can be transformed to everywhere divergent (for $|z| = 1$) using some phase correction $\tilde{c}_n = e^{i\,\phi_n}\, c_n$. Now let us turn to expansion (2) of σ and remark that Riemann–Lebesgue lemma can be interperted in the folowing way.

Lemma 1. *Given an absolutely continuous measure σ_{ac},*

$$\widehat{\sigma}_{ac}(n) \to 0 \quad \textit{as} \quad n \to \infty.$$

Definition 2. We call *Menshov–Rajchman measure*, a singular measure satisfying $\widehat{\mu}(n) \to 0$, $n \to \infty$. We denote as $\mathscr{R}$ the class of all measures of such kind.

Evidently any discrete measure σ_d is never of Menshov–Rajchman type since its Fourier transform $\widehat{\sigma}_d(n)$ is a Bohr almost periodic sequence. At the same time, it is easy to see that the singular Cantor–Lebesgue measure μ_{CL} supported on the standard 1/3-Cantor set enjoys the property $\widehat{\mu}_{\mathrm{CL}}(3^k n) = \widehat{\mu}_{\mathrm{CL}}(n)$, which is explained by fractal symmetry of the Cantor set. Thus, $\mu_{\mathrm{CL}} \notin \mathscr{R}$. Modifying the construction of μ_{CL} Menshov [11] provided the first example of singular measure in the class $\mathscr{R}$. Further, Neder [12] proved that any Menshov–Rajchman measure cannot be a mixture of discrete and continuous component, and then Wiener [25] extended this result and showed that the Fourier coefficients of any continuous measure μ converge to zero in average, and any set $\{n\colon |\widehat{\mu}(n)| > b > 0\}$ has zero density in $\mathbb{Z}$. Littlewood [8] found a singular probability measure σ with the rate of decay

$$\widehat{\sigma}(n) = O\big(|n|^{-c}\big), \qquad c > 0.$$

Then Wiener and Wintner [26] obtained a stronger result demonstrating that the exponent c can be arbitrary close to $1/2$, but the approach proposed by the authors generates a measure σ that depends on $c = 1/2 - \alpha$, $\alpha > 0$. Soon after this work Schaeffer [22] using the idea of Riesz products proved the existence of a singular σ with

$$\widehat{\sigma}(n) = O\big(r(|n|) \cdot |n|^{-1/2}\big)$$

for any given increasing sequence $r(n) \to \infty$, $n \to \infty$. In particular, σ satisies

$$\widehat{\sigma}(n) = O(|n|^{-1/2+\varepsilon}) \quad \text{for any} \quad \varepsilon > 0.$$

Ivashev-Musatov [6] got a further improvement of Schaeffer's result. He found a set of singular measures with sub-$|n|^{-1/2}$ rate of correlation decay satisfying $\widehat{\sigma}(n) = O(\rho(n) \cdot |n|^{-1/2+\varepsilon})$ with $\rho(n) \to 0$ but $\rho(n) \gg |n|^{-\varepsilon}$ for any $\varepsilon > 0$. Following [26] let us denote $\kappa(\sigma)$ the infinum of real γ's such that $\widehat{\sigma}(n) = O(|n|^{\gamma})$. In a series of works [19, 20, 21] Salem introduced an approach that helps to see, in particular, explicit examples of singular distributions with the property $\kappa(\sigma) = -1/2$.

Definition 3. Let us call singular measures on $\mathbb{T}^d$ (respectively, $\mathbb{R}^d$) satisfying the condition $\widehat{\sigma}(n) = O(|n|^{-d/2+\varepsilon})$ for any $\varepsilon > 0$, *measures of Schaeffer–Salem type.*

In this note we discover that Schaeffer–Salem measures appear as spectral measures for a class of group actions with invariant measure. Let us consider a measure preserving invertible transformation $T\colon\ X \to X$ of the standard Lebesgue space $(X, \mathcal{B}, \mu)$ and define *Koopman operator* on the space $H = L^2(X, \mathcal{B}, \mu)$,

$$\hat{T}\colon\ H \to H\colon\ f(x) \mapsto f(Tx).$$

Clearly, $\hat{T}$ is a unitary operator in a separable Hilbert space H, hence, it is characterized up to a unitary equivalence by the pair $(\sigma_{(T)}, M(z))$, where $\sigma_{(T)}$ is the *measure of maximal spectral type* and $M(z)$ is the *multiplicity function.* Of course, two transformations which are spectrally isomorphic, need not be isomorphic as dynamical systems. For example, all Bernoulli shifts have Lebesgue spectrum of infinite multiplicity but they are dinstiguished by entropy. And, in fact, it is a hard problem far from complete understanding to classify all pairs $(\sigma_{(T)}, M(z))$ that can appear as spectral invariants of a measure preserving transformation (see [7, 10]). Further, given an element $f \in L^2(X, \mathcal{B}, \mu)$, the *spectral measure* $\sigma_f \in \mathscr{M}$ on $\mathbb{T} \simeq [0, 1)$ is uniquely defined by the relation

$$\widehat{\sigma}_f(n) = \int e^{-2\pi i\, xn}\, d\sigma(x) = R_f(-n) \overset{def}{=} \left\langle \hat{T}^{-n} f,\, f \right\rangle.$$

It is easy to see that $\sigma_f \ll \sigma_{(T)}$.

2. Dynamical systems generating Schaeffer–Salem measures

In this section we define actions of the groups $\mathbb{Z}^d$ and $\mathbb{R}^d$ with invariant measure generating spectral measures of Schaeffer–Salem type for a dense set of function on the phase space. Without loss of generality we concentrate our attention on the case of $\mathbb{Z}^d$-actions. Consider a nested sequence of lattices Γ_n in the acting group $G = \mathbb{Z}^d$ having $G_{n+1} \subset G_n$, and let $M_n = G/\Gamma_n$ be the corresponding sequence of homogeneous spaces linked with a natural

projection $\pi_n\colon M_{n+1} \to M_n$ mapping $a+\Gamma_{n+1}$ onto the point $a+\Gamma_n$. Let us also fix a Fölner sequence of Γ_n-fundamental domains U_n, where

$$G = \bigcup_{\gamma\in\Gamma_n} (\gamma + U_n) \quad \text{and} \quad \gamma_1 + U_n \cap \gamma_2 + U_n = \varnothing, \quad \text{whenever} \quad \gamma_1 \neq \gamma_2.$$

To simplify understanding of the construction let us consider a particular case $\Gamma_n = h_n\mathbb{Z}^d$, $h_{n+1} = q_n h_n$, $q_n \in \mathbb{Z}$, and let U_n be rectangles $U_n = [0, h_n)^{\times d}$. Now let us introduce the maps $\phi_n\colon\ M_{n+1} \to M_n$,

$$\phi_n(\gamma + u) = u + \alpha_{n,\gamma}, \qquad \gamma \in \Gamma_n/\Gamma_{n+1}, \quad u \in U_n.$$

The idea of ϕ_n can be explained as flows. Take a pair M_n and M_{n+1}. We consider M_{n+1} as a union (up to a boundary in the case of $\mathbb{R}^d$) of finitely many domains $\gamma + U_n$, where $\gamma \in \Gamma_n/\Gamma_{n+1}$. Next we note that each domain $\gamma + U_n$ is mapped by π_n in 1-to-1 way to the manifold M_n, and then we define ϕ_n in such a way that $\gamma + U_n$ after descending to M_n is "rotated" to some fixed vector $\alpha_{n,\gamma}$. In a sense the boundaries of the domains $\gamma+U_n$ contains all the discontinunity points of the map ϕ_n. For a given point $a \in M_{n+1}$ let us define $\boldsymbol{\gamma}_n(a)$ to be the unique element $\gamma \in \Gamma_n/\Gamma_{n+1}$ such that $a \in \gamma + U_n$. Let μ_n be the normalized shift invariant measure on the space M_n. Then $\mu_{n+1}(\phi_n^{-1}(A)) \equiv \mu_n(A)$. Further, let us define X to be the inverse limit of the spaces M_n,

$$X = \bigl\{x = (x_1, x_2, \ldots, x_n, \ldots)\colon\ x_n \in M_n,\ \phi_n(x_{n+1}) = x_n\bigr\}.$$

The space X is endowed both with the weak topology and with the structure of probability space $(X, \mathcal{B}, \mu)$, where σ-algebra $\mathcal{B}$ is generated by the cylindric sets $B_{n,A} = \{x\colon\ x_n \in A\}$, where A is a Borel set in M_n, and μ is the measure extending μ_n. Notice that $\tilde{\pi}_n^*\mu = \mu_n$, where $\tilde{\pi}_n(x) = x_n$.

Finally let us define an action of the group G on the space $(X, \mathcal{B}, \mu)$. Assume now that $q_{n+1} > 2q_n$. Fix $t \in G$. It is to check using Borel–Cantelli lemma that the probability

$$\mu\bigl\{x \in X\colon\ \exists\, n_0(x) \quad \forall\, n \geq n_0(x) \quad \boldsymbol{\gamma}_n(x_n) = \boldsymbol{\gamma}_n(t + x_n)\bigr\} = 1,$$

hence, for the points x of such kind (belonging to the set above) we can define T^t by the rule $(T^t x)_n = t + x_n$ for $n \geq n_0(x)$. For indexes $n < n_0(x)$ the coordinates $(T^t x)_n$ are recovered using the fundamental equation $\phi_n(x_{n+1}) = x_n$. The following lemma directly follows from the definition (see also [16] for the careful examination of the case $G = \mathbb{Z}$).

Lemma 4. *The maps T^t provide an action of the group G on the space $(X, \mathcal{B}, \mu)$ by measure preserving transformations.*

We call the class of systems provided by this construction *systems of iceberg type*. It can be easilily seen that this class is analogous to the general rank one actions (see [3, 15]). The spectral measures of rank one transformation and flows are given by a certain class of generalized Riesz products [1, 2, 14] and for the study of spectral type for systems of iceberg type one can also use a variation of Riesz products.

Let us remark that the $\mathbb{Z}$-action of iceberg type with $q_n = 2$, $(\alpha_{n,0}, \alpha_{n,1}) = (0, h_n/2)$, is identical to the classical Morse transformation (see [24] for the discussion of arithmetic properties of Morse systems).

Theorem 5. *Let $\alpha_{n,\gamma}$ be a family of independent random variables, uniformely distributed on finite sets Γ_n/Γ_{n+1}. Then for a certain sequence $q_n \to \infty$ and for a set of (cylindric) functions f dense in $L^2(X,\mu)$ the spectral measures σ_f satisfy almost surely the condition*

$$\widehat{\sigma}_f(t) = O(|t|^{-1/2+\varepsilon}) \tag{3}$$

for any $\varepsilon > 0$. Moreover, $\widehat{\sigma}_f \notin L^2(G)$, and $\widehat{\sigma}_f \in L^p(G)$ for $p > 2$.

In the case $G = \mathbb{Z}$ the proof of the next theorem is given in [16].

The most important effect, underlying the proof of the theorem, is the universal character of the rate of correlation decay $R_f(-t) = \langle T^{-t} f,\, f\rangle = \widehat{\sigma}_f(t)$ for this class of dynamic systems. As we will see, it follows easily from the calculation of expectations $\mathsf{E}|R_f(-t)|^2$ for random variables $R_f(-t)$, but reconstruction of σ_f from the sigma $\widehat{\sigma}_f(t)$ is a complicated problem, and even for the random family of dynamical systems of theorem 5 we cannot say whether measures σ_f are almost surely singular?

In paper [18] Ryzhikov proves the existence of transformations preserving an infinite measure satisfying the estimate $\kappa(\sigma_f) \le -1/2$ for a dense set of elements f.

Proof of theorem 5. Without loss of generality let us assume that $d = 1$. As it is mentioned above the proof of theorem 5 is rather simple. It is based on an elementary and common phenomenon which hides more special combinatorial properties of the correlation sequence $R_f(t)$ as well as properties of the spectral measure σ_f. Indeed, a universal estimate of the expectation $\mathsf{E}|R_f(t)|^2$ is established for our class of systems:

$$\mathsf{E}|R_f(t)|^2 \le r(n) \cdot t^{-1}$$

for $h_{n-1} \ll t \le h_n$, where $r(n)$ is a steadily increasing sequence: $r(n) \ll h_n$. Let f be a cylindric function depending only on x_{n_0}, i.e. $f(x) = f_{n_0}(x_{n_0})$, It is enough to consider just the *correlation functions*

$$R_n^\circ(t) = \frac{1}{h_n^d} \int_{M_n} f_n(x-t)\, \overline{f_n(x)}\, d\mu_n,$$

where f_n is the lift of the function f to the manifold M_n. It is also sufficient to consider a particular set of values $t = h_{n-1} \cdot s \in \Gamma_{n-1}$, $s \in \mathbb{Z}^d \setminus \{0\}$. Such values of t are characterized by the following properties. Whenever $t \in \Gamma_{n-1}$ the shift transformation $x \mapsto t + x$ preserves the partition of the manifold M_n to the sets $\gamma + U_{n-1}$. Then given a cylindric function f with zero mean, $\int f\, d\mu = 0$, we see that

$$\mathsf{E} R_n^\circ(t) = \frac{1}{q_{n-1}} \sum_{\gamma \in \Gamma_{n-1}/\Gamma_n} \mathsf{E}\left\langle \rho_{\alpha_{n-1,\gamma}} f_{n-1},\, \rho_{\alpha_{n-1,\gamma+s}} f_{n-1} \right\rangle = 0,$$

since $\alpha_{n-1,\gamma}$ are $\alpha_{n-1,\gamma+s}$ are independent. Developping this technique we establish a reccurent property

$$\mathsf{E}|R_n^\circ(t)|^2 \sim \frac{1}{q_{n-1}} \cdot \mathsf{E}_{\mu_{n-1}} \mathsf{E}|R_{n-1}^\circ|^2,$$

and the estimate

$$\mathsf{E}\|R_n^\circ\|^2 \le \mathsf{E}\|R_{n-1}^\circ\|^2 + h_n \cdot h_{n-1}^{-1} \mathsf{E}\|R_{n-1}^\circ\|^2 \le 2\, \mathsf{E}\|R_{n-1}^\circ\|^2 \cdot (1 + o(1)) = O(2^n),$$

hence,

$$\mathsf{E}|R_n^\circ(t)|^2 = O(q_{n-1}^{-1} \cdot 2^n h_{n-1}^{-1}) = O(h_n^{-1} \cdot 2^n) = O(t^{-1} \cdot 2^n),$$

where $t \le h_n$ and the proof is finished. □

It follows directly from theorem 5 that $\sigma_f * \sigma_f \ll \lambda$, where λ is the normalized invariant measure on $\widehat{G}$. Thus, our observation is connected to the open question due to Banach, — *"Does there exist a $\mathbb{Z}$-action with invariant probability measure having Lebesgue spectrum of multiplicity one?"* — since the spectral multiplicity for almost every action in theorem 5 equals one (and for *all* systems of iceberg type we have $M(z) \le 4$).

Open questions and hypotheses.

(i) Dynamical systems in theorem 5 are almost surely of singular spectrum
(ii) $\kappa(\sigma_f) = -1/2$ for all systems of iceberg type, i.e. for any choice of the parameters $\alpha_{n,\gamma}$, where f is a cylindric function
(iii) Can we find a speed of Fourier coefficient decay $\widehat{\sigma}(t) = O(|t|^{-1/2}\,\rho(t))$, that can be reached in the class of *all* singular measures, but *impossible* for measures of maximal spectral type generated by measure preserving transformations? (cf. question (iv))
(iv) Given a measure of Salem type, $\kappa(\sigma) = -1/2$, is it posible to reach any speed of decay of type $\widehat{\sigma}(t) = O(|t|^{-1/2}\,\rho(t))$ with $\forall\, \varepsilon > 0 \quad |t|^{-\varepsilon} \le \rho(t) \le |t|^{\varepsilon}$ just multiplying by some density $p(x) \in L^1(\sigma)$?
(v) Is it true that all $\mathbb{Z}^d$- and $\mathbb{R}^d$-actions of iceberg type have singular spectrum?

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